Advanced Probability : End-Semester Exam

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Submit solutions via Moodle by 18th December 1:30 PM.

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Please write and sign the following declaration on your answer script first :

I have not received, I have not given, nor will I give or receive, any assistance to another student taking this exam, including discussing the exam with other students. The solution to the problems are my own and I have not copied it from anywhere else. I have used only class notes and the notes of D. Panchenko, R. Durrett and M. Krishnapur.

Attempt any four questions only. Each question carries 10 points. If you attempt more than four questions, the first four answers will be evaluated.

1. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. uniform random vectors in the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$. Let $B_1, \ldots, B_k$ be Borel subsets of D for a fixed k. Let $N_{i,n} := |\{X_1, \ldots, X_n\} \cap B_i|, 1 \leq i \leq k$ be the number of points $X_1, \ldots, X_n$ falling inside B_i. Consider the vector $N_n = (N_{1,n}, \ldots, N_{k,n}), n \geq 1$. Is there a vector μ_n and scalar $\sigma_n \geq 0$ such that
$$\frac{N_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0, C),$$
for some matrix C ? If yes, then find μ_n, σ_n and C as well.

2. Define $C(A, B) := |A \cap B|$ for $A, B \subset \mathbb{R}^d$, bounded Borel subsets. Show that given bounded Borel subsets $A_1, \ldots, A_k, k \geq 1$, there exists a multivariate Normal random vector $X = (X_1, \ldots, X_k)$ with mean 0 and Covariance matrix C given by $C(i, j) = |A_i \cap A_j|, 1 \leq i \leq j \leq k$. Further, if B, C are bounded Borel subsets, set $A_1 = B, A_2 = C, A_3 = B \cap C, A_4 = B \cup C$ and define the vector X as above for $k = 4$. Show that $X_4 = X_1 + X_2 - X_3$ a.s..

3. Given $0 < p < 1$, consider i.i.d. random variables $X_i, i \geq 1$ such that $\mathbb{P}(X_i = 1) = p = 1 - \mathbb{P}(X_i = -1)$ and let $S_0 = 0, S_n = \sum_{i=1}^{n} X_i$. Let $a, b \in \mathbb{Z}$ with $a \leq -1, b \geq 1$. Define
$$\tau = \min\{k : S_k \in \{a, b\}\}.$$

(a) Show that $\mathbb{P}(\tau \geq n) \to 0$ as $n \to \infty$.

(b) Compute $\mathbb{E}[\tau]$.

4. Let $\mathcal{B}_n, n \geq 0$ be a filtration and $Z_n, n \geq 0$ be a predictable sequence of bounded random variables w.r.t. $\mathcal{B}_n$.

   (a) If $(X_n, \mathcal{B}_n), n \geq 0$ is a super-martingale and $Z_n \geq 0$, show that $Y_n = Z_0 X_0 + \sum_{k=1}^n Z_k(X_k - X_{k-1})$ is a super-martingale and $\mathbb{E}[Y_n] \geq \mathbb{E}[X_n]$.

   (b) If $(X_n, \mathcal{B}_n), n \geq 0$ is a martingale, show that $Y_n = Z_0 X_0 + \sum_{k=1}^n Z_k(X_K - X_{k-1})$ is a martingale.

   (c) If τ is a stopping time and $(X_n, \mathcal{B}_n), n \geq 0$ is a super-martingale , show that $X_{\tau \wedge n}$ is a super-martingale.

5. Let there be r red balls and b blue balls in an urn and $r, b \geq 1$. At step $n = 1, 2, \ldots$ we pick a ball from the urn at random and replace it with c balls of the same colour. Let R_n, B_n be the number of red and blue balls after $n (\geq 1)$ steps respectively. Show that $\frac{R_n}{R_n + B_n}, n \geq 0$ is a martingale with respect to the natural filtration and compute $\lim_{n \to \infty} \mathbb{E}[\frac{R_n}{R_n + B_n}]$.

6. Let $(X_n, Z_n), n \geq 1$ be i.i.d. random vectors such that X_n are Poisson random variables with mean 1 and Z_n is a sequence of mean 0 random variables with finite variance. We are not assuming that X_n, Z_n are independent. Is there a_n, b_n such that

$$a_n \left(\sum_{i=1}^n (X_i \log i + Z_i) - b_n \right) \xrightarrow{d} N(0,1)?$$